Asymptotic analysis of a particle system with mean-field interaction

Anatoli Manita [*],

Faculty of Mathematics and Mechanics,

Moscow State University, 119992, Moscow, Russia.

E-mail: manita@mech.math.msu.su

Vadim Shcherbakov[†],

CWI, Postbus 94079, 1090 GB,

Amsterdam, The Netherlands

E-mail: V.Shcherbakov@cwi.nl

Abstract

We study a system of N interacting particles on $\mathbf{Z}$. The stochastic dynamics consists of two components: a free motion of each particle (independent random walks) and a pair-wise interaction between particles. The interaction belongs to the class of *mean-field* interactions and models a *rollback synchronization* in asynchronous networks of processors for a distributed simulation. First of all we study an empirical measure generated by the particle configuration on $\mathbf{R}$. We prove that if space, time and a parameter of the interaction are appropriately scaled (hydrodynamical scale), then the empirical measure converges weakly to a deterministic limit as N goes to infinity. The limit process is defined as a weak solution of some partial differential equation. We also study the long time evolution of the particle system with fixed number of particles. The Markov chain formed by individual positions of the particles is not ergodic. Nevertheless it is possible to introduce *relative* coordinates and prove that the new Markov chain is ergodic while the system as a whole moves with an asymptotically constant mean speed which differs from the mean drift of the free particle motion.

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1   Introduction

We study an interacting particle system which models a set of processors performing parallel simulations. The system can be described as follows. Consider $N \geq 2$ particles moving in $\mathbf{Z}$. Let $x_i(t)$ be the position at time t of the $i-$th particle, $1 \leq i \leq N$. Each particle has three clocks.

The first, the second and the third clock, attached to the $i-$th particle, ring at the moments of time given by a mutually independent Poisson processes $\Pi_{i,\alpha}$, $\Pi_{i,\beta}$ and Π_{i,μ_N} with intensities α, β and μ_N correspondingly. These triples of Poisson processes for different indexes are also independent. Consider a particle with index i. If the first attached clock rings, then the particle jumps to the nearest right site: $x_i \rightarrow x_i + 1$, if the second attached clock rings, then the particle jumps to the nearest left site: $x_i \rightarrow x_i - 1$. At moments when the third attached clock rings a particle with index j is chosen with probability $1/N$ and if $x_i > x_j$, then the $i-$th particle is relocated: $x_i \rightarrow x_j$. It is supposed that all these changes occur immediately.

The type of the interaction between the particles is motivated by studying of probabilistic models in the theory of parallel simulations in computer science ([16], [22, 23] and [6, 11]). The main peculiarity of the models is that a group of processors performing a large-scale simulation is considered and each processor does a specific part of the task. The processors share data doing simulations therefore their activity must be synchronized. In practice, this synchronization is achieved by applying a so-called *rollback* procedure which is based on a massive message exchange between different processors (see [2, Sect. 1.4 and Ch. 8]). One says that $x_i(t)$ is a *local time* of the i-th processor while t is a real (absolute) time. If we interpret the variable $x_i(t)$ as an amount of job done by the processor i till the time moment t, then the interaction described above imitates this synchronization procedure. Note that from a point of view of general stochastic particle systems the interaction between the particles is essentially non-local.

We are interested in the analysis of asymptotic behaviour of this particle system. First of all we consider the situation as the number of particles goes to infinity. For every finite N and t we can define an empirical measure generated by the particle configuration. It is a point measure with atoms at integer points. An atom at a point k equals to a proportion of particles with coordinate k at time t. It is convenient in our case to consider an empirical tail function corresponding to the measure. It means that we consider $\xi_{x,N}(t) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\,(x_i(t) \geq x)$ the proportion of particles having coordinates not less than $x \in \mathbf{R}$. The problem is to find an appropriate time scale t_N and a sequence of interaction parameters μ_N to obtain a non-trivial limit dynamics of the process $\xi_{N,[xN]}(t_N)$ as $N \rightarrow \infty$. The cases $\alpha \neq \beta$ and $\alpha = \beta$ require different scaling of time and the interaction constant μ_N. We prove that there exist non-trivial limit deterministic processes in both cases as N goes to infinity if we rescale time and the interaction constant as $t_N = tN$, $\mu_N = \mu/N$ in the first case and as $t_N = tN^2$, $\mu_N = \mu/N^2$ in the second case respectively. The processes are defined as weak solutions of some partial differential equations (PDE). It should be noted that the PDE relating to the zero drift situation is a famous *Kolmogorov–Petrowski–Piscounov*-equation (KPP-equation, [9]). This result was announced in [19].

Another issue we address in the paper is studying of the long time evolution of the particle system with fixed number of particles. It is easy to see that the Markov chain $x(t) = \{x_i(t),\ i = 1,\dots,N\}$, $t \geq 0$, is not ergodic. Nevertheless the particle system possesses some relative stabil-

ity. We introduce new coordinates $y_i(t) = x_i(t) - \min_j x_j(t)$, $i = 1, \ldots, N$, and prove that the countable Markov chain $y(t) = \{y_i(t), i = 1, \ldots, N\}$, $t \geq 0$, is ergodic and converges exponentially fast to its stationary distribution. Therefore the system of stochastic interacting particles possesses some relative stability. We show also that the center of mass of the system moves with an asymptotically constant speed. It appears that due to the interaction between the particles this speed differs from the mean drift of the free particle motion.

It should be noted that the choice of the interaction may vary depending on a situation. Various modifications of the model can be considered and similar results can be obtained using the same methods. We have chosen the described model just for the sake of concreteness.

Probabilistic models for parallel computation considered before by other authors. The paper [16] deals with a model consisting of two interacting processors ($N = 2$). It contains a rigorous study of the long-time behavior of the system and formulae for some performance characteristics. Unfortunately, there are not too many mathematical results about multi-processor models ([12, 13, 14, 22, 23]). Usually mathematical components of these papers have a form of preparatory considerations before some large numerical simulation. The paper [6] is of special interest because it rigorously studies a behavior of some model of parallel computation with N processor units in the limit $N \to \infty$. A stochastic dynamics of [6] is different from the dynamics studied in the present paper and main results of [6] concern a so-called thermodynamical limit. The authors prove that in the limit the evolution of the system can be described by some integro-differential equation. In the present study we propose a model which dynamics is easy from the point of view of numerical simulations and, at the same time, provides us with a new probabilistic interpretation of some important PDEs including the classical KPP-equation.

The paper is organised as follows. We formally define the particle system, introduce some notation and formulate the main results in Section 2. Sections 3 and 4 contain the proofs of the main results. In Section 5 we discuss solutions of the limiting equations.

2   The model and main results

Formally, the process $x(t) = \{x_i(t), i = 1, \ldots, N\}$, describing positions of the particles, is a continuous time countable Markov chain taking values in $\mathbf{Z}^N$ and having the following generator

$$G_N g(x) = \sum_{i=1}^{N} \left(\alpha \left(g \left(x + e_i^{(N)} \right) - g(x) \right) + \beta \left(g \left(x - e_i^{(N)} \right) - g(x) \right) \right) +$$

$$+ \sum_{i=1}^{N} \sum_{j \neq i} \left(g \left(x - e_i^{(N)} (x_i - x_j) \right) - g(x) \right) I_{\{x_i > x_j\}} \frac{\mu_N}{N}, \quad (1)$$

where $x = (x_1, \ldots, x_N) \in \mathbf{Z}^N$, $g : \mathbf{Z}^N \to \mathbf{R}$ is a bounded function, $e_i^{(N)}$ is a N-dimensional vector with all zero components except $i-$th which equals to 1, $I_{\{x_i > x_j\}}$ is an indicator of the set $\{x_i > x_j\}$.

Define

$$\xi_{N,k}(t) = \frac{1}{N} \sum_{i=1}^{N} I_{\{x_i(t) \geq k\}}, \quad k \in \mathbf{Z}. \quad (2)$$

The process $\xi_N(t) = \{\xi_{N,k}(t), \, k \in \mathbf{Z}\}$ is a Markov one with a state space H_N the set of all non-negative and nonincreasing sequences $z = \{z_k, k \in \mathbf{Z}\}$ such that $z_k \in \{l/N, \, l = 0, 1, \ldots, N\}$ for any $k \in \mathbf{Z}$ and

$$\lim_{k \to -\infty} z_k = 1, \qquad \lim_{k \to +\infty} z_k = 0.$$

The generator of the process $\xi_N(t)$ is given by the following formula

$$L_N f(z) = N \sum_k ((f(z + e_k/N) - f(z))\alpha(z_{k-1} - z_k) + (f(z - e_k/N) - f(z))\beta(z_k - z_{k+1}))$$

$$\quad (3)$$

$$+ N \mu_N \sum_{l < k} (f(z - (e_{l+1} + \ldots + e_k)/N) - f(z))(z_k - z_{k+1})(z_l - z_{l+1}),$$

where $e_i, i \in \mathbf{Z}$ is an infinite dimensional vector with all zero components except $i-$th which equals to 1, $f : H_N \to \mathbf{R}$ is a bounded function.

2.1   Hydrodynamical behavior of the particle system

Denote $\zeta_{N,x}(t) = \xi_{N,[Nx]}(t), \ x \in \mathbf{R}$. The process $\zeta_N(t)$ takes values in $H = H(\mathbf{R})$ the set of all non-negative right continuous with left limits nonincreasing functions having the following limits

$$\lim_{x \to -\infty} \psi(x) = 1, \ \lim_{x \to \infty} \psi(x) = 0.$$

Denote by $S(\mathbf{R})$ the Schwartz space of infinitely differentiable functions such that for all $m, n \in \mathbf{Z}_+$

$$\|f\|_{m,n} = \sup_{x \in \mathbf{R}} |x^m f^{(n)}(x)| < \infty.$$

Recall that $S(\mathbf{R})$ equipped with a natural topology given by seminorms $\| \cdot \|_{m,n}$ is a Frechet space ([18]).

Define for every $h \in H$ a functional

$$(h, f) = \int_{\mathbf{R}} h(x) f(x) dx, \ f \in S(\mathbf{R}),$$

on the Schwartz space $S(\mathbf{R})$. The following bound yields that for each $h \in H$ $(h, \cdot)$ is a continuous linear functional on $S(\mathbf{R})$

$$|(h, f)| \leq \int_{\mathbf{R}} |f(x)| \frac{1 + x^2}{1 + x^2} dx \leq \pi(\|f\|_{\infty} + \|x^2 f\|_{\infty}) \equiv \pi(\|f\|_{0,0} + \|x^2 f\|_{2,0}),$$

where $\| \cdot \|_{\infty}$ is the supremum norm. Thus the set of functions $H(\mathbf{R})$ is naturally embedded into the space of all continuous linear functionals on $S(\mathbf{R})$, namely into the space $S'(\mathbf{R})$ of tempered distributions. We will interpret $\zeta_N(t)$ as a stochastic process taking its values in the space $S'(\mathbf{R})$.

There are two reasons for embedding $H(\mathbf{R})$ into $S'(\mathbf{R})$ and considering the $S'(\mathbf{R})$-valued processes. The first reason is that due to some nice topological properties of $S'(\mathbf{R})$ we can use in Section 3 many powerful results from the theory of weak convergence of probability distributions on topological vector fields. And, secondly, the choice of $S'(\mathbf{R})$ as a state space is convenient from the point of view of possible future study of stochastic fluctuation fields around the deterministic limits obtained in our main theorem 2.1.

In the sequel we mainly deal with the strong topology (*s.t.*) on $S'(\mathbf{R})$ (see Section A.1). From now we fix some $T > 0$ and consider ζ_N as a random element in a Skorokhod space $D([0, T], S'(\mathbf{R}))$ of all mappings of $[0, T]$ to $(S'(\mathbf{R}), s.t.)$ that are right continuous and have left-hand limits in the strong topology on $S'(\mathbf{R})$. Note that $(S'(\mathbf{R}), s.t.)$ is not a metrisable topological space therefore it is not evident how to define the Skorokhod topology on the space $D([0, T], S'(\mathbf{R}))$. To do this we follow [15] and refer to Section A.1.

Now we are able to consider probability distributions of the processes $(\zeta_N(tN^a), t \in [0, T])$, $a = 1, 2$, as probability measures on a measurable space $\left(D([0, T], S'(\mathbf{R})), \mathcal{B}_{D([0,T],S'(\mathbf{R}))} \right)$ where $\mathcal{B}_{D([0,T],S'(\mathbf{R}))}$ is a corresponding Borel σ-algebra. It was proved in [7] that $\mathcal{B}_{D([0,T],S'(\mathbf{R}))} = \mathcal{C}_{D([0,T],S'(\mathbf{R}))}$, where $\mathcal{C}_{D([0,T],S'(\mathbf{R}))}$ is a σ-algebra of cylindrical subsets.

Consider two following Cauchy problems

$$\begin{aligned} u_t(t, x) &= -\lambda u_x(t, x) + \mu \left(u^2(t, x) - u(t, x) \right), \\ u(0, x) &= \psi(x) \end{aligned} \tag{4}$$

and

$$u_t(t, x) = \gamma u_{xx}(t, x) + \mu(u^2(t, x) - u(t, x)),\tag{5}$$
$$u(0, x) = \psi(x)$$

where u_t, u_x and u_{xx} are first and second derivatives of u with respect to t and x. Notice that the equation (5) is a particular case of the famous *Kolmogorov–Petrowski–Piscounov*-equation (KPP-equation, [9]). We will deal with weak solutions of the equations (4) and (5) in the sense of Definition 2.1.

Fix $T > 0$ and denote by $C_{0,T}^\infty = C_0^\infty([0, T] \times \mathbf{R})$ the space of infinitely differentiable functions with finite support and equal to zero for $t = T$.

Definition 2.1 (i) *The bounded measurable function $u(t, x)$ is called a weak (or generalized) solution of the Cauchy problem (4) in the region $[0, T] \times \mathbf{R}$, if the following integral equation holds for any function $f \in C_{0,T}^\infty$*

$$\int_0^T \int_{\mathbf{R}} u(t, x)(f_t(t, x) + \lambda f_x(t, x)) + \mu u(t, x)(1 - u(t, x))f(t, x))dxdt$$

$$+ \int_{\mathbf{R}} u(0, x)f(0, x)dx = 0$$

(ii) *The bounded measurable function $u(t, x)$ is called a weak (or generalized) solution of the Cauchy problem (5) in the region $[0, T] \times \mathbf{R}$, if the following integral equation holds for any function $f \in C_{0,T}^\infty$*

$$\int_0^T \int_{\mathbf{R}} u(t, x)(f_t(t, x) + \gamma f_{xx}(t, x)) + \mu u(t, x)(1 - u(t, x))f(t, x))dxdt$$

$$+ \int_{\mathbf{R}} u(0, x)f(0, x)dx = 0$$

In Subsection 3.5 we will show that the both of Cauchy problems (4) and (5) have *unique weak solutions* in the sense of Definition 2.1. Here we want just to mention that this problem is not trivial. Indeed, the equation (4) is an example of a quasilinear first order partial differential equation. It is known that in a general case such type of equations might have more than one weak solution and it is only possible to guarantee uniqueness of the solution which satisfies to the so-called entropy

condition. The most general form of this condition was introduced by Kruzhkov in [10], where he also proved his famous uniqueness theorem. Fortunately, in our particular case of the equation (4) the situation is quite simple due to simplicity of characteristics, they are given by the straight lines $x(t) = \lambda t + C$, do not intersect with each other and do not produce the shocks. Detailed discussions of the problem of uniqueness for equations (4) and (5) are presented in Subsection 3.5.

The first theorem we are formulating describes the evolution of the system at the hydrodynamical scale.

Theorem 2.1 *Assume that an initial particle configuration $\xi_N(0) = \{\xi_{N,k}(0), \ k \in \mathbf{Z}\}$ is such that for any function $f \in S(\mathbf{R})$*

$$\lim_{N \to \infty} \frac{1}{N} \sum_k \xi_{N,k}(0) f(k/N) = \int_{\mathbf{R}} \psi(x) f(x) dx, \qquad (6)$$

where $\psi \in H(\mathbf{R})$.

(i) *If $\alpha - \beta = \lambda \neq 0$ and $\mu_N = \mu/N$, then the sequence $\{Q_{N,\lambda}^{(T)}\}_{N=2}^\infty$ of probability distributions of random processes $\{\zeta_N(tN), \ t \in [0, T]\}_{N=2}^\infty$ converges weakly as $N \to \infty$ to the probability measure $Q_\lambda^{(T)}$ on $D([0, T], S'(\mathbf{R}))$ supported by a trajectory $u(t, x)$, which is a unique weak solution of the equation (4) with the initial condition $u(0, x) = \psi(x)$ and as a function of x $u(t, \cdot) \in H(\mathbf{R})$, for any $t \geq 0$.*

(ii) *If $\alpha = \beta = \gamma > 0$, $\mu_N = \mu/N^2$, then the sequence $\{Q_{N,\gamma}^{(T)}\}_{N=2}^\infty$ of probability distributions of random processes $\{\zeta_N(tN^2), \ t \in [0, T]\}_{N=2}^\infty$ converges weakly as $N \to \infty$ to the probability measure $Q_\gamma^{(T)}$ on $D([0, T], S'(\mathbf{R}))$ supported by a trajectory $u(t, x)$, which is a unique weak solution of the equation (5) with the initial condition $u(0, x) = \psi(x)$ and as a function of x $u(t, \cdot) \in H(\mathbf{R})$, for any $t \geq 0$.*

2.2 Long time behavior of the particle system with the fixed number of particles

The number of particles is fixed in this section. Consider the following stochastic process $y(t) = (y_1(t), \ldots, y_N(t))$, where

$$y_i(t) = x_i(t) - \min_j x_j(t).$$

Note that $x_k - x_l = y_k - y_l$ for any pair k, l.

It is easy to see that $y(t)$ is a continuous time Markov chain on the state space

$$\Gamma = \bigcup_k \Gamma_k \subset \mathbf{Z}_+^N,$$

where $\Gamma_k := \{(z_1, \ldots, z_{k-1}, 0, z_{k+1}, \ldots, z_N) : z_j \in \mathbf{Z}_+\}$.

Theorem 2.2 *The Markov chain $(y(t), t \geq 0)$ is ergodic and converges exponentially fast to its stationary distribution*

$$\sum_{y \in \Gamma} |P(y(t) = y) - \pi(y)| \leq C_1 \exp(-C_2 t)$$

uniformly in initial distributions of $y(0)$.

3   Proof of Theorem 2.1

3.1   Plan of the proof

The proof of the convergence uses the next well-known general idea (see, for example, [21, § 5]). Let $\{a_n\}$ be a sequence in some Hausdorff topological space and assume that $\{a_n\}$ satisfies to the following two properties: (a) for any subsequence of $\{a_n\}$ there is a converging subsequence (this property is called a *sequential compactness*); (b) $\{a_n\}$ contains at most one limit point. Then the sequence $\{a_n\}$ has a limit.

In our situation the role of $\{a_n\}$ is played by the sequences $\{Q_{N,\lambda}^{(T)}\}_{N=2}^{\infty}$ and $\{Q_{N,\gamma}^{(T)}\}_{N=2}^{\infty}$. Our proof consists of the following steps.

Step 1. We fix an arbitrary $T > 0$ and prove that the sequences of probability measures $\{Q_{N,\lambda}^{(T)}\}_{N=2}^{\infty}$ and $\{Q_{N,\gamma}^{(T)}\}_{N=2}^{\infty}$ are tight. We use the Mitoma theorem ([15]) and apply martingale techniques widely used in the theory of hydrodynamical limits of interacting particle systems ([4, 8]).

It is important to note that if a topological space $\mathcal{V}$ is not metrisable then, generally speaking, the tightness of a family of distributions on $\mathcal{V}$ does not imply a sequential compactness (see, for example, [3, V. 2, § 8.6]). So, in general, the above property (a) does not follow directly from the step 1. But in our concrete case $\mathcal{V} = D([0, T], S'(\mathbf{R}))$ we can proceed as follows. It was shown in [7] that any compact subset of $D([0, T], S'(\mathbf{R}))$ is metrisable. Due to this property we can apply the theorem from [21, Th. 2, § 5] which states that (under assumption of metrisability of compact subsets) the tightness of a family of measures implies its sequential compactness. All this justifies the next step.

Step 2. We show that a measure that is a limit of some subsequence of the sequence $\{Q_{N,\lambda}^{(T)}\}_{N=2}^{\infty}$ (or $\{Q_{N,\gamma}^{(T)}\}_{N=2}^{\infty}$) is supported by the weak solutions of the partial differential equation (4) (or, correspondingly, (5)). Then we note that each of the equations (4) and (5) has a unique weak solution (Subsection 3.5). This gives the above property (b).

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3.2 Technical lemmas

We start with some bounds which will be used throughout the proof. Denote

$$R_f(z) = \frac{1}{N} \sum_k f(k/N) z_k,$$

for $z \in H_N$ and $f \in S(\mathbf{R})$.

Lemma 3.1 (i) *If $\alpha \neq \beta$ and $\mu_N = \mu/N$, then for any $z \in H_N$*

$$|L_N R_f(z)| \leq \frac{C}{N}, \tag{7}$$

and

$$N\left(L_N R_f^2(z) - 2R_f(z) L_N R_f(z)\right) = O\left(\frac{1}{N}\right). \tag{8}$$

(ii) *If $\alpha = \beta$ and $\mu_N = \mu/N^2$, then for any $z \in H_N$*

$$|L_N R_f(z)| \leq \frac{C}{N^2}, \tag{9}$$

and

$$N^2\left(L_N R_f^2(z) - 2R_f(z) L_N R_f(z)\right) = O\left(\frac{1}{N^2}\right). \tag{10}$$

In both cases $C = C(f, \alpha, \beta, \mu)$.

Proof of Lemma 3.1. We will prove the bounds (7) and (8), the other ones can be proved similarly. We start with the bound (7). Using the equations

$$R_f(z + e_k/N) - R_f(z) = \frac{f(k/N)}{N^2},$$

$$R_f(z - e_k/N) - R_f(z) = -\frac{f(k/N)}{N^2}$$

we get that for every $z \in H_N$

$$L_N R_f(z) = \frac{1}{N} \sum_k z_k (\beta f((k-1)/N) - (\alpha + \beta) f(k/N) + \alpha f((k+1)/N))$$

$$- \frac{\mu}{N^2} \sum_k f(k/N) z_k (1 - z_k).$$

For any function $f \in S(\mathbf{R})$ consider its upper Darboux sum

$$U_N^+(f) = \frac{1}{N} \sum_{k \in \mathbf{Z}} \max_{y \in [k/N, (k+1)/N]} f(y). \tag{11}$$

Since $U_N^+(f) \to \int f(x)\,dx$ as $N \to \infty$, the sequence $\{U_N^+(f)\}_{N=1}^{\infty}$ is bounded in N for any fixed f. We have uniformly in $z \in H_N$

$$
\begin{aligned}
|L_N R_f(z)| \;\leq\; & \frac{1}{N} \left(\sum_k |\alpha(f((k+1)/N) - f(k/N)) + \beta(f(k/N) - f((k-1)/N))| \right) \\
& + \; \frac{\mu}{N} \left(\frac{1}{N} \sum_k |f(k/N)| \right) \\
\leq \; & \frac{1}{N} \left(|\alpha - \beta| U_N^+ (|f_x|) + \mu U_N^+ (|f|) \right),
\end{aligned}
$$

where $f_x = df(x)/dx$. So the bound (7) is proved.

Let us prove the bound (8). Note that $L_N = L_N^{(0)} + L_N^{(1)}$, where

$$L_N^{(0)} f(z) = N \sum_k (f(z + e_k/N) - f(z))\alpha(z_{k-1} - z_k) +$$

$$N \sum_k (f(z - e_k/N) - f(z))\beta(z_k - z_{k+1})$$

and

$$L_N^{(1)} f(z) = \mu \sum_{l<k} (f(z - (e_{l+1} + \ldots + e_k)/N) - f(z))(z_k - z_{k+1})(z_l - z_{l+1}).$$

Using the equations

$$R_f^2(z + e_k/N) - R_f^2(z) = \left(2R_f(z) + \frac{f(k/N)}{N^2} \right) \frac{f(k/N)}{N^2},$$

$$R_f^2(z - e_k/N) - R_f^2(z) = - \left(2R_f(z) - \frac{f(k/N)}{N^2} \right) \frac{f(k/N)}{N^2},$$

one can obtain that for any $z \in H_N$

$$
\begin{aligned}
L_N^{(0)} R_f^2(z) &= \alpha N \sum_k (R_f^2(z + e_k/N) - R_f^2(z))(z_{k-1} - z_k) \\
&+ \beta N \sum_k (R_f^2(z - e_k/N) - R_f^2(z))(z_k - z_{k+1}) \\
&= 2R_f(z) \frac{\alpha}{N} \sum_k f(k/N)(z_{k-1} - z_k) + \frac{\alpha}{N^3} \sum_k f^2(k/N)(z_{k-1} - z_k) \\
&- 2R_f(z) \frac{\beta}{N} \sum_k f(k/N)(z_k - z_{k+1}) + \frac{\beta}{N^3} \sum_k f^2(k/N)(z_k - z_{k+1}) \\
&= 2R_f(z) L_N^{(0)} R_f(z) + O\left(\frac{1}{N^2}\right).
\end{aligned}
\tag{12}
$$

Direct calculation gives that for any function $g_{k,j}(z) = z_k z_j, k < j$, on H_N

$$
L_N^{(1)} g_{k,j}(z) = -\frac{\mu}{N}(z_k z_j(1 - z_k) + z_k z_j(1 - z_j)) + \frac{\mu}{N^2} z_j(1 - z_k).
\tag{13}
$$

Using this formula we get that for any $z \in H_N$

$$
\begin{aligned}
L_N^{(1)} R_f^2(z) &= -2R_f(z) \frac{\mu}{N^2} \sum_k f(k/N) z_k(1 - z_k) \\
&+ \frac{2\mu}{N^4} \sum_{k<j} f(k/N)f(j/N) z_j(1 - z_k) + \frac{\mu}{N^4} \sum_k f^2(k/N) z_k(1 - z_k) \\
&= 2R_f(z) L_N^{(1)} R_f(z) + O\left(\frac{1}{N^2}\right).
\end{aligned}
\tag{14}
$$

Summing the formulas (12) and (14) we obtain

$$
L_N R_f^2(z) = 2R_f(z) L_N R_f(z) + O\left(\frac{1}{N^2}\right).
\tag{15}
$$

Lemma 3.1 is proved.

3.3 Tightness

We will make all considerations for the part (**i**) ($\alpha \neq \beta$ and $\mu = \mu/N$) of the theorem. All reasonings and conclusions are valid for the part (**ii**) ($\alpha = \beta$ and $\mu = \mu/N^2$) of the theorem with evident changes. We will denote by $P_{N,\lambda}^{(T)}$ the probability distribution on the path space $D([0, T], H_N)$ corresponding to the process $\xi_N(t)$ and by $E_{N,\lambda}^{(T)}$ the expectation with respect to this measure.

Theorem 4.1 in [15] (see also Section A.2) yields that tightness of the sequence of $\{Q_{N,\lambda}^{(T)}\}_{N=2}^{\infty}$ will be proved if we prove the same for a sequence of distributions of one-dimensional projection $\{(\zeta_N(tN), f), t \in [0, T]\}_{N=2}^{\infty}$ for every $f \in S(\mathbf{R})$. So fix $f \in S(\mathbf{R})$ and consider the sequence of distributions of the processes $(\zeta_N(tN), f)$, $t \in [0, T]$. Note that the probability distribution of a process $(\zeta_N(tN), f)$ is a probability measure on $D([0, T], \mathbf{R})$ the Skorokhod space of real-valued functions.

By definition of the process $\zeta_N(tN)$ we have that

$$(\zeta_N(tN), f) = \sum_k \xi_{N,k}(tN) \int\limits_{k/N}^{(k+1)/N} f(x)dx.$$

It is easy to see that

$$(\zeta_N(tN), f) = R_f(\xi_N(tN)) + \phi_N(t),$$

where the random process $\phi_N(t)$ is bounded $|\phi_N(t)| < C(f)/N$ for any $t \geq 0$ and sufficiently large N. Therefore it suffices to prove tightness of the sequence of distributions of random processes $\{R_f(\xi_N(tN)), t \in [0, T]\}_{N=2}^{\infty}$.

Introduce two random processes

$$W_{f,N}(t) = R_f(\xi_N(tN)) - R_f(\xi_N(0)) - N \int\limits_0^t L_N R_f(\xi_N(sN))ds, \tag{16}$$

and

$$V_{f,N}(t) = (W_{f,N}(t))^2 - \int\limits_0^t Z_{f,N}(s)ds,$$

where

$$Z_{f,N}(s) = N \left(L_N R_f^2(\xi_N(sN)) - 2R_f(\xi_N(sN))L_N R_f(\xi_N(sN)) \right). \tag{17}$$

It is well known (Theorem 2.6.3 in [4] or Lemma A1.5.1 in [8]) that the processes $W_{f,N}(t)$ and $V_{f,N}(t)$ are martingales.

The bound (7) yields that

$$\left| N \int\limits_{\tau}^{\tau+\theta} L_N R_f(\xi_N(sN))ds \right| \leq C\theta \quad (a.s.)$$

for any time moment τ. Thus the sequence of probability distributions of random processes $\{N \int_0^t L_N R_f(\xi_N(sN))ds, t \in [0, T]\}_{N=2}^{\infty}$ is tight by Theorems A.2 and A.3 from Appendix.

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The bound (8) yields that

$$E_{N,\lambda}^{(T)}(W_{f,N}(\tau+\theta)-W_{f,N}(\tau))^2 = E_{N,\lambda}^{(T)}\left(\int_\tau^{\tau+\theta} Z_{f,N}(s)ds\right) \leq \frac{C\theta}{N}. \tag{18}$$

for any stopping time $\tau \geq 0$ since $V_{f,N}(s)$ is martingale. Using this estimate and Chebyshev inequality we obtain that the sequence of probability distributions of martingales $\{W_{f,N}(t),\ t \in [0,T]\}_{N=2}^\infty$ is also tight by Theorem A.3. Thus the sequence of probability distributions of the processes $\{R_f(\xi_N(tN)),\ t \in [0,T]\}_{N=2}^\infty$ is tight by the equation (16) and the assumption (6) and, hence, the sequence of probability measures $\{Q_{N,\lambda}^{(T)}\}_{N=2}^\infty$ is tight by Theorem 4.1 in [15].

3.4 Characterization of a limit point

We are going to show now that there is a unique limit point of the sequence $\{Q_{N,\lambda}^{(T)}\}_{N=2}^\infty$ and this limit point is supported by trajectories which are weak solutions of the partial differential equation (4) in the sense of Definition 2.1.

Let $f(s,x) \in C_{0,T}^\infty$ and denote

$$R_f(t,\xi_N(tN)) = \frac{1}{N}\sum_k \xi_{N,k}(tN)f(t,k/N),$$

Define as before two random processes

$$W'_{f,N}(t) = R_f(t,\xi_N(tN)) - R_f(0,\xi_N(0)) - \int_0^t (\partial/\partial s + NL_N)\,R_f(s,\xi_N(sN))ds,$$

and

$$V'_{f,N}(t) = (W'_{f,N}(t))^2 - \int_0^t Z'_{f,N}(s)ds,$$

where

$$Z'_{f,N}(s) = N\left(L_N R_f^2(s,\xi_N(sN)) - 2R_f(s,\xi_N(sN))L_N R_f(s,\xi_N(sN))\right).$$

By Lemma A1.5.1 in [8] the processes $W'_{f,N}(t)$ and $V'_{f,N}(t)$ are martingales. It is easy to see that

$$W'_{f,N}(t) = (\zeta(tN), f) - \int_0^t (\zeta_N(sN), f_s + \lambda f_x + \mu f)ds - (\zeta(0), f)$$

$$+ \mu \int_0^t R_f(s, \xi^2(sN))ds + O\left(\frac{1}{N}\right). \quad (19)$$

We are going to approximate the nonlinear term in (19) by some quantities making sense in the space of generalised functions since we treat the processes distributions as probability measures on a space $D([0, T], S'(\mathbf{R}))$. Let $\varkappa \in C_0^\infty(\mathbf{R})$ be a non-negative function such that $\int_R \varkappa(y)\, dy = 1$. Denote $\varkappa_\varepsilon(y) = \varkappa(y/\varepsilon)/\varepsilon$, for $0 < \varepsilon \le 1$ and let $(\varkappa_\varepsilon * \varphi(s))(x) = \int_{\mathbf{R}} \varkappa_\varepsilon(x - y)\varphi(y, s)dy$ be a convolution of a generalised function $\varphi(s, \cdot)$ with the test function $\varkappa_\varepsilon(y)$.

Lemma 3.2 *The following uniform estimate holds*

$$\left|R_f(s, \xi^2(sN)) - ((\varkappa_\varepsilon * \zeta_N(sN))^2, f)\right| \le F_1(\varepsilon) + F_2(\varepsilon N)$$

where the functions F_1 and F_2 do not depend on ξ and s and

$$\lim_{\varepsilon \downarrow 0} F_1(\varepsilon) = \lim_{r \to +\infty} F_2(r) = 0.$$

Proof. For definiteness we assume that $\varkappa(x) = 0$ for $x \in (-\infty, -1 - \delta') \cup (1 + \delta', \infty)$ for some positive δ'. It is easy to see that if $x \in [k/N, (k+1)/N)$ for some k, then

$$(\varkappa_\varepsilon * \zeta_N(sN))(x) = \sum_j \xi_{N,j}(sN) \int_{j/N}^{(j+1)/N} \varkappa_\varepsilon(x - y)dy$$

$$= \frac{1}{N}\sum_j \varkappa_\varepsilon((k - j)/N)\xi_{N,j}(sN) + g_1(N, \varepsilon, x, \xi_N(sN))$$

where the function $g_1(N, \varepsilon, x, \xi_N(sN))$ can be bounded as follows

$$|g_1(N, \varepsilon, x, s, \xi_N(sN))| \le \sum_j \int_{j/N}^{(j+1)/N} \left|\varkappa_\varepsilon(x - y) - \varkappa_\varepsilon\left(\frac{k-j}{N}\right)\right| dy$$

$$\le \frac{1}{N}\sum_m 2 \max_{w \in [(m-1)/N, m/N]} \frac{1}{\varepsilon^2}\left|\varkappa'\left(\frac{w}{\varepsilon}\right)\right| \cdot \frac{1}{N}$$

$$= \frac{2}{(N\varepsilon)^2}\sum_m \max_{v \in [(m-1)/(N\varepsilon), m/(N\varepsilon)]} |\varkappa'(v)| = \frac{2\, U_{N\varepsilon}^+(|\varkappa'|)}{N\varepsilon}$$

(see the (11) for the notation U^+). Note that if ε is fixed then $U^+_{N\varepsilon}(|\varkappa'|) = O(1)$ as $N \to \infty$.

This representation implies that

$$((\varkappa_\varepsilon * \zeta_N(sN))^2, f) = \frac{1}{N} \sum_k f(k/N) \left(\frac{1}{N} \sum_j \varkappa_\varepsilon((k-j)/N)\xi_{N,j}(sN) \right)^2 + O\left(\frac{1}{\varepsilon N}\right)$$

Therefore

$$|R_f(s, \xi^2(sN)) - ((\varkappa_\varepsilon * \zeta_N(sN))^2, f)| \leq J_{f,s}(\delta', \varepsilon, N) + K(\varepsilon, N) + O\left(\frac{1}{\varepsilon N}\right),$$

where

$$J_{f,s}(\delta', \varepsilon, N) = \frac{2}{N} \sum_k |f(s, k/N)| \frac{1}{N} \sum_{j:|j-k|<(1+\delta')\varepsilon N} \varkappa_\varepsilon((k-j)/N)|\xi_{N,k}(sN) - \xi_{N,j}(sN)|$$

and

$$K(\varepsilon, N) = Const \cdot \left| \frac{1}{N} \sum_m \varkappa_\varepsilon \left(\frac{m}{N} \right) - 1 \right|.$$

Evidently, $K(\varepsilon, N) = Const \cdot \left| \frac{1}{N\varepsilon} \sum_m \varkappa \left(\frac{m}{N\varepsilon} \right) - 1 \right|$ and, hence, $K(\varepsilon, N)$ tends to 0 in the limit "ε is fixed, $N \to \infty$". So we can include $K(\varepsilon, N)$ into $F_2(\varepsilon N)$.

Consider now the term $J_{f,s}(\delta', \varepsilon, N)$. Using monotonicity of trajectories we obtain that for any j such that $|j - k| < (1 + \delta')\varepsilon N$

$$|\xi_{N,k}(sN) - \xi_{N,j}(sN)| \leq \xi_{N,k-[(1+\delta')\varepsilon N]}(sN) - \xi_{N,k+[(1+\delta')\varepsilon N]}(sN).$$

Thus we have that

$$J_{f,s}(\delta', \varepsilon, N) \leq \frac{2}{N} \sum_k |f(s, k/N)|(\xi_{N,k-[(1+\delta')\varepsilon N]}(sN) - \xi_{N,k+[(1+\delta')\varepsilon N]}(sN)).$$

Integrating by parts we get the following bound

$$J_{f,s}(\delta', \varepsilon, N) \leq \frac{2}{N} \sum_k (|f(s, (k + [(1+\delta')\varepsilon N])/N)| - |f(s, (k - [(1+\delta')\varepsilon N])/N)|) \, \xi_{N,k}(sN)$$

$$\leq \frac{2}{N} \sum_k |f(s, (k + [(1+\delta')\varepsilon N])/N) - f(s, (k - [(1+\delta')\varepsilon N])/N)|$$

$$\leq Const \cdot MD \cdot (1 + \delta')\varepsilon,$$

where $M = \max_{x,s} |f_x(s, x)|$, D is a diameter of supp $f_x(s, x)$.

Note that the last inequality is uniform in trajectories. Lemma 3.2 is proved.

Lemma 3.3 *For every $\delta > 0$*

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} P_{\lambda,N}^{(T)} \left(\left| \int\limits_0^T \left(R_f(s, \xi^2(sN)) - ((\varkappa_\varepsilon * \zeta_N(sN))^2, f) \right) ds \right| > \delta \right) = 0.$$

The proof of this lemma is omitted because it is a direct consequence of Lemma 3.2.

It is easy to see that for any $f \in C_{0,T}^\infty$ a map $F_{f,T,\varepsilon}(\varphi) : D([0,T], S') \to \mathbf{R}_+$ defined by

$$F_{f,T,\varepsilon}(\varphi) = \left| \int\limits_0^T \left((\varphi(s), f_s + \lambda f_x + \mu f) - \mu((\varkappa_\varepsilon * \varphi(s))^2, f) \right) ds + (\varphi(0), f) \right|$$

is continuous, therefore for any $\delta > 0$ the set $\{\varphi \in D([0,T], S') : F_{f,T,\varepsilon}(\varphi) > \delta\}$ is open and hence

$$\limsup_{\varepsilon \to 0} Q_\lambda^{(T)} \left(\varphi : F_{f,T,\varepsilon}(\varphi) > \delta \right) \leq \limsup_{\varepsilon \to 0} \liminf_{N \to \infty} Q_{N,\lambda}^{(T)} \left(\varphi : F_{f,T,\varepsilon}(\varphi) > \delta \right),$$

where $Q_\lambda^{(T)}$ is a limit point of the sequence $\{Q_{N,\lambda}^{(T)}\}_{N=2}^\infty$. Obviously that

$$Q_{N,\lambda}^{(T)} \left(\varphi : F_{f,T,\varepsilon}(\varphi) > \delta \right) \leq P_{N,\lambda}^{(T)} \left(\sup_{t \leq T} |W_{f,N}'(t)| > \delta/2 \right) \tag{20}$$

$$+ P_{N,\lambda}^{(T)} \left(\left| \int\limits_0^T (R_f(s, \xi^2(sN)) - ((\varkappa_\varepsilon * \zeta_N(sN))^2, f) ds \right| > \delta/2 \right).$$

It is easy to see that the bound (8) obtained in Lemma 3.1 for the process $Z_{f,N}(s)$ is also valid for the process $Z_{f,N}'(s)$, therefore for any t

$$E_{N,\lambda}^{(T)}(W_{f,N}'(t))^2 = E_{N,\lambda}^{(T)} \left(\int\limits_0^t Z_{f,N}'(s)ds \right) \leq \frac{Ct}{N},$$

since $V_{f,N}'(t)$ is a martingale. Kolmogorov inequality implies that for any $\delta > 0$

$$P_{N,\lambda}^{(T)} \left(\sup_{t \leq T} |W_{f,N}'(t)| \geq \delta \right) \leq \delta^{-2} E_{N,\lambda}^{(T)}(W_{f,N}'(T))^2 = \delta^{-2} E_{N,\lambda}^{(T)} \left(\int\limits_0^T Z_{f,N}'(s)ds \right) \leq \frac{CT}{N\delta^2}. \tag{21}$$

The second term in (20) vanishes to zero by Lemma 3.3 as $N \to \infty$ and $\varepsilon \to 0$. Therefore for any $f \in C_{0,T}^\infty$ and $\delta > 0$

$$\limsup_{\varepsilon \to 0} Q_\lambda^{(T)} \left(\varphi : F_{f,T,\varepsilon}(\varphi) > \delta \right) = 0. \tag{22}$$

Let us prove that we can replace the convolution in $F_{f,T,\varepsilon}$ by its limit which is well defined with respect to the measure $Q_\lambda^{(T)}$. First of all we note that for any $C > 0$ $B_C(\mathbf{R})$ the set of measurable functions h such that $\|h\|_\infty \le C$ is a closed subset of $S'(\mathbf{R})$ in both strong and weak topology. Indeed, consider a sequence of functions $g_n \in B_C$, $n \ge 1$ and assume this sequence converges in S' to some tempered distribution $G \in S'$. We are going to show that this generalized function is determinated by some measurable function bounded by the same constant C. It is easy to see that for every $n \ge 1$

$$\left| \int_{\mathbf{R}} g_n(x) f(x) dx \right| \le C \|f\|_{L^1}, \ f \in S,$$

where $\| \cdot \|_{L^1}$ is a norm in L^1 the space of all integrable functions. So the limit linear functional G on S is also continuous in L^1-norm

$$|G(f)| \le C \|f\|_{L^1}, \ f \in S.$$

The space S is a linear subspace of L^1 therefore by Hahn-Banach Theorem (Theorem III-5, [18]) the linear functional G can be extended to a continuous linear functional $\tilde{G}$ on L^1 with the same norm and such that $\tilde{G}|_S = G$. Using the theorem about the general form of a continuous linear functional on L^1 ([18]) we obtain that

$$G(f) = \int_{\mathbf{R}} g(x) f(x) dx, \ f \in L^1,$$

where g is a measurable bounded function. Obviously that $\|g\|_\infty \le C$.

Obviously that for any $N \ge 2$ and fixed $t \in [0, T]$ we have that $Q_{N,\lambda}^{(T)} (\varphi(t, \cdot) \in B_1) = 1$, where $\varphi(t, \cdot) = \varphi(t)$ is a coordinate variable on $D([0, T], S')$. Therefore if some subsequence $\{Q_{N',\lambda}^{(T)}\}$ of $\{Q_{N,\lambda}^{(T)}\}_{N=2}^\infty$ converges weakly to a limit point $Q_\lambda^{(T)}$, then for any fixed $t \in [0, T]$

$$Q_\lambda^{(T)} (\varphi(t) \in B_1) \ge \limsup_{N' \to \infty} Q_{N',\lambda}^{(T)} (\varphi(t) \in B_1) = 1, \tag{23}$$

since B_1 is closed. Next lemma gives an important property of the convolution on the set of bounded functions L^∞.

Lemma 3.4 *Fix $\varphi \in L^\infty$. Then*
1) for any $0 \le s \le T$

$$((\varkappa_\varepsilon * \varphi(s))^2, f) - \int_{\mathbf{R}} \varphi^2(s, x) f(x) \, dx \to 0 \qquad (\varepsilon \to 0)$$

2) for any $0 \leq t \leq T$

$$\int\limits_0^t \left\{ ((\varkappa_\varepsilon * \varphi(s))^2, f) - \int_{\mathbf{R}} \varphi^2(s, x) f(x) \, dx \right\} ds \to 0 \qquad (\varepsilon \to 0).$$

Proof of Lemma 3.4.

$$\left| ((\varkappa_\varepsilon * \varphi(s))^2, f) - \int_{\mathbf{R}} \varphi^2(s, x) f(x) \, dx \right| \leq \int_{\mathbf{R}} \left| \left(((\varkappa_\varepsilon * \varphi(s))(x) \right)^2 - \varphi^2(s, x) \right| |f(x)| \, dx$$

$$\leq 2\|\varphi\|_\infty \int_{\mathbf{R}} \left| ((\varkappa_\varepsilon * \varphi(s))(x) - \varphi(s, x) \right| |f(x)| \, dx$$

To get the last inequality we used the identity $a^2 - b^2 = (a + b)(a - b)$ and the fact that $\|\varkappa_\varepsilon *$ $\varphi\|_\infty \leq \|\varphi\|_\infty$. To finish the proof it suffices to apply a well-known result about convergence of $(\varkappa_\varepsilon * \varphi)(s, \cdot)$ to $\varphi(s)$ in L^1_{loc}. This proves the first statement of the lemma. To get the second statement we use again the boundedness of φ and $\varkappa_\varepsilon * \varphi$ and apply the Lebesgue theorem. Lemma 3.4 is proved.

On the set $\operatorname{supp} Q_\lambda^{(T)}$ we can define a functional

$$F_{f,T}^0(\varphi) = \int\limits_0^T (\varphi(s), f_s + \lambda f_x + \mu f) - \mu(\varphi^2(s), f)) ds + (\varphi(0), f).$$

The equation (23) and Lemma 3.4 yield that for any $\varphi \in \operatorname{supp} Q_\lambda^{(T)}$ $F_{f,T,\varepsilon}(\varphi) \to F_{f,T}^0(\varphi)$ as $\varepsilon \to 0$. This implies that for any $\delta_1 > 0$

$$Q_\lambda^{(T)} \left(\varphi : |F_{f,T,\varepsilon}(\varphi) - F_{f,T}^0(\varphi)| > \delta_1 \right) \to 0 \qquad (\varepsilon \to 0).$$

Combining this with (22) we get that a limit point of the sequence $\{Q_{N,\lambda}^{(T)}\}_{N=2}^\infty$ is concentrated on the trajectories $\varphi(t, \cdot)$, $t \in [0, T]$, taking values in the set of regular bounded functions and satisfying the following integral equation

$$\int\limits_0^T (\varphi(s), f_s + \lambda f_x + \mu f) - \mu(\varphi^2(s), f)) ds + (\varphi(0), f) = 0$$

for any $f \in C_{0,T}^\infty$. It means that each such a trajectory is a weak solution of the equation (4) in the sense of Definition 2.1.

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3.5   Uniqueness of a weak solution

The first order equation.   Using the method of Theorem 1 in the celebrated paper [17] of Oleinik we will show here that for any measurable bounded initial function $\psi(x)$ there might be at most one weak solution of the equation (4) in the sense of Definition 2.1 and no entropy condition is required.

Let $u(t,x)$ and $v(t,x)$ be two weak solutions of the equation (4) in the region $[0,T] \times \mathbf{R}$ with the same initial condition ψ (not necessarily from H). Definition 2.1 implies that

$$\int\limits_0^T \int\limits_{\mathbf{R}} ((u(t,x) - v(t,x))(f_t(t,x) + \lambda f_x(t,x) + \mu(1 - u(t,x) - v(t,x))f(t,x))dxdt = 0, \quad (24)$$

for any $f \in C_{0,T}^\infty$. Consider the following sequence of equations

$$f_t(t,x) + \lambda f_x(t,x) + g_n(t,x)f(t,x) = F(t,x), \tag{25}$$

with any infinitely differentiable function F equal to zero outside of a certain bounded region, lying in the half-plane $t \geq \delta_1 > 0$, where δ_1 is an arbitrary small number. The functions $g_n(t,x)$ are uniformly bounded for all $x,t, n \geq 1$ and converges in L_{loc}^1 to the function $g(t,x) = \mu(1 - u(t,x) - v(t,x))$ as $n \to \infty$. The solution $f_n(t,x) \in C_{0,T}^\infty$ of the equation (25) is given by the following formula (formula (2.8) in [17])

$$f_n(t,x) = \int\limits_T^t F(s, x + \lambda(s-t)) \exp\left\{ \int\limits_t^s g_n(\tau, x + \lambda(\tau - t))d\tau \right\} ds.$$

Equation (24) yields that

$$\int\limits_{\mathbf{R_+}} \int\limits_{\mathbf{R}} (u(t,x) - v(t,x))F(t,x)dxdt = \int\limits_{\mathbf{R_+}} \int\limits_{\mathbf{R}} (g(t,x) - g_n(t,x))f(t,x))dxdt. \tag{26}$$

The right side of (26) is arbitrary small for sufficiently large n and, since the left side of (26) does not depend on n, so it is equal to zero. Therefore $u = v$, since F is arbitrary.

An *existence* of a weak solution of the equation (4) follows from the general theory for quasi-linear equations of the first order (for example, Theorem 8 in [17]). In the particular case of the equation (4) it is possible to obtain an explicit formula for a weak solution. First of all we note that the Cauchy problem

$$u_t(t,x) = -\lambda u_x(t,x) + \mu(u^2(t,x) - u(t,x)), \qquad u(0,x) = \psi(x), \tag{27}$$

has a unique classical solution, if $\psi \in C^1(\mathbf{R})$ and there is an explicit formula for this solution. Indeed, using substitution $u^\circ(t, x) = u(t, x - \lambda t)$ we transform the equation (27) into the equation

$$u_t^\circ(t, x) = -\mu u^\circ(t, x)(1 - u^\circ(t, x)), \qquad u^\circ(0, x) = \psi(x).$$

Considering x as a parameter we obtain an ordinary differential equation which is solvable and the solution is given by:

$$u^\circ(t, x) = \frac{\psi(x)e^{-\mu t}}{1 - \psi(x) + \psi(x)e^{-\mu t}}. \tag{28}$$

So, if $\psi \in C^1(\mathbf{R})$, then a unique classical solution of the equation (27) is given by the following formula

$$u(t, x) = \frac{\psi(x + \lambda t)e^{-\mu t}}{1 - \psi(x + \lambda t) + \psi(x + \lambda t)e^{-\mu t}}. \tag{29}$$

If we approximate any measurable bounded function g in L_{loc}^1 by a sequence of smooth functions $\{g_n, n \geq 1\}$, then the sequence of corresponding weak solutions $\{u_n(t, x), n \geq 1\}$, where $u_n(t, x)$ is defined by the formula (29) with $\psi = g_n$, converges in L_{loc}^1 to the weak solution of the equation with initial condition g by Theorem 11 in [17] or Theorem 1 in [10]. It is easy to show by direct calculation that the L_{loc}^1–limit of the sequence $\{u_n(t, x), n \geq 1\}$ is given by the same formula (29) with $\psi = g$.

The formula (29) yields that if $\psi \in H(\mathbf{R})$, then $u(t, \cdot) \in H(\mathbf{R})$ as a function of x for any fixed $t \geq 0$. If a function $u(t, x)$ is a weak solution of the equation (27), then this function is differentiable at a point (t, x) iff the initial condition $\psi(y)$ is differentiable at the point $y = x + \lambda t$.

KPP-equation. The equation (5) is a quasilinear parabolic equations of the second order and is a particular case of the famous KPP-equation in [9]. It is known that there exists a unique weak solution $u(t, x)$ of the problem

$$u_t(t, x) = \gamma u_{xx}(t, x) + \mu(u^2(t, x) - u(t, x)), \qquad u(0, x) = \psi(x),$$

for any bounded measurable initial function ψ, this solution is in fact a unique classical solution and if $\psi \in H(\mathbf{R})$, then $u(t, \cdot) \in H(\mathbf{R})$ as a function of x for any fixed t. We refer to the paper [17] for more details.

4 System with fixed number of particles

In this section we deal with the situation "N is fixed, $t \to \infty$".

4.1 Proof of Theorem 2.2

Let $\sigma(y, w)$ be the rate of transition from the state $y = (y_1, \ldots, y_N) \in \Gamma$ to the state $w = (w_1, \ldots, w_N) \in \Gamma$ for the Markov $y(t)$ chain. Define $\sigma(y) = \sum_{w \neq y} \sigma(y, w)$. From definition of the particle system it follows that

$$\sigma(y) = (\alpha + \beta)N + \frac{\mu_N}{N} \sum_{(i,j)} I_{\{y_i > y_j\}}.$$

Since $\sum_{(i,j)} I_{\{y_i > y_j\}} \leq N(N - 1)/2$ we have uniformly in $y \in \Gamma$

$$\sigma_{*,N} \leq \sigma(y) \leq \sigma_N^* \tag{30}$$

with $\sigma_{*,N} = (\alpha + \beta)N$ and $\sigma_N^* = (\alpha + \beta)N + \mu_N(N - 1)/2$. A discrete time Markov chain $\{Y(n), \, n = 0, 1, \ldots\}$ on the state space S with transition probabilities

$$p(y, w) \equiv \mathsf{P}\{Y(n + 1) = w \,|\, Y(n) = y\} = \begin{cases} \dfrac{\sigma(y, w)}{\sigma(y)}, & y \neq w \\ 0, & y = w, \end{cases} \tag{31}$$

is an embedded Markov chain of the continuous time Markov chain $(y(t), \, t \geq 0)$.

Theorem 2.2 is a consequence of the following statement.

Lemma 4.1 *The Markov chain $\{Y(n), \, n = 0, 1, \ldots\}$ is irreducible, aperiodic and satisfies to the Doeblin condition: there exist $\varepsilon > 0$, $m_0 \in \mathbf{N}$ and finite set $A \subset \Gamma$ such that*

$$\mathsf{P}\{Y(m_0) \in A \,|\, Y(0) = Y_0\} \geq \varepsilon, \tag{32}$$

for any $Y_0 \in S$. Therefore this Markov chain is ergodic ([5]).

Proof of Lemma 4.1. We are going to show that condition (32) holds with $A = \{(0, \ldots, 0)\}$, $m_0 = N$,

$$\varepsilon = \left(\frac{\min(\alpha, \beta, \mu_N/N)}{\sigma_N^*} \right)^N > 0.$$

The transition probabilities of the Markov chain $\{Y(n), \, n = 0, 1, \ldots\}$ are uniformly bounded from below in the following sense: if a pair of states (z, v) is such that $\sigma(z, v) > 0$ (or, equivalently, $p(z, v) > 0$) then (31) implies that

$$p(z, v) > \min(\alpha, \beta, \mu_N/N) / \sigma_N^*. \tag{33}$$

So to prove (32) we need only to show that for any y there exists a sequence of states

$$v^0 = y, \quad v^1, \quad v^2, \quad \ldots, \quad v^N = (0, \ldots, 0) \tag{34}$$

which can be subsequently visited by the Markov chain $\{Y(n), n = 0, 1, \ldots\}$. The last means that i.e. $p(v^{n-1}, v^n) > 0$ for every $n = 1, \ldots, N$ and hence

$$\mathsf{P}\{Y(N) \in A \mid Y(0) = y\} \geq \prod_{n=1}^{N} p(v^{n-1}, v^n) \geq \left(\frac{\min\left(\alpha, \beta, \mu_N/N\right)}{\sigma_N^*}\right)^N$$

as a consequence of the uniform bound (33).

   To prove existence of the sequence (34) let us assume first that $y = (y_1, \ldots, y_N) \neq 0$. Choose and fix some r such that $y_r = \max_i y_i > 0$. Denote by

$$n_0 = \#\{j : y_j = 0\}$$

the number of left-most particles. Let the right-most particle y_r move n_0 steps to right:

$$v^n - v^{n-1} = e_r^{(N)}, \qquad n = 1, \ldots, n_0.$$

This can be done by using of jumps to the nearest right state. So $Y(n_0) = v^{n_0}$ has exactly n_0 particles at 0 and $N - n_0$ particles out of 0. Denote by $i_{n_0+1} < i_{n_0+2} < \cdots < i_N$ indices of particles with $v_{j_a}^{n_0} > 0$, $a = n_0 + 1, \ldots, N$. Let now the Markov chain Y transfer each of these particles to 0:

$$v^a - v^{a-1} = -v_{i_a}^{n_0} e_{i_a}^{(N)}, \qquad a = n_0 + 1, \ldots, N.$$

It is possible due to transitions provided by the interaction.

   To complete the proof we need to consider the case $y = (y_1, \ldots, y_N) = 0$. It is quite easy:

$$v^1 = e_1^{(N)}, \quad v^2 = 2e_1^{(N)}, \quad v^3 = 3e_1^{(N)}, \quad \ldots, \quad v^{N-1} = (N-1)e_1^{(N)}, \quad v^N = 0.$$

Proof of the lemma is over.

   Denote by $\pi^Y = \left(\pi^Y(y), y = (y_1, \ldots, y_N) \in \Gamma\right)$ a unique stationary distribution of the Markov chain $\{Y(n), n = 0, 1, \ldots\}$.

   The proof of Theorem 2.2 is now easy. First of all let us show that the uniform bound $\sigma(y) \geq \sigma_{*,N}$ implies existence of a stationary distribution for the Markov chain $y(t)$. Indeed, it is easy to check that if π^Y is the stationary distribution of the embedded Markov chain Y and Q is the infinitesimal matrix for the chain $y(t)$, then a vector with positive components $s = (s(w), w \in S)$ defined as

$$s(w) = \frac{\pi^Y(w)}{\sigma(w)},$$

satisfies to the equation $sQ = 0$. So for existence of a stationary distribution of the chain $y(t)$ it is sufficient to show $\sum_{w \in S} s(w) < +\infty$. It is easy to check the last condition:

$$\sum_{w \in S} s(w) = \sum_{w \in S} \frac{\pi^Y(w)}{\sigma(w)} \leq \frac{1}{\sigma_{*,N}} \sum_{w \in S} \pi^Y(w) = \frac{1}{\sigma_{*,N}}.$$

Therefore the continuous-time Markov chain $(y(t),\, t \geq 0)$ has a stationary distribution $\pi = (\pi(y),\, y \in \Gamma)$ of the following form

$$\pi(y) = \frac{\dfrac{\pi^Y(y)}{\sigma(y)}}{\displaystyle\sum_{w \in \Gamma} \frac{\pi^Y(w)}{\sigma(w)}}.$$

Denote $p_{yw}(t) = \mathsf{P}\{y(t) = w \mid y(0) = 0\}$. The next step is to prove that the continuous-time Markov chain $y(t)$ is ergodic. To do this we show that the following Doeblin condition holds: *for some $j_0 \in S$ there exists $h > 0$ and $0 < \delta < 1$ such that $p_{ij_0}(h) \geq \delta$ for all $i \in S$.* It is well-known ([5]) that this condition implies ergodicity and moreover

$$|p_{ij}(t) - \pi(y)| \leq (1 - \delta)^{[t/h]}.$$

Let $\tau_k,\, k \geq 0$, be the time of stay of the Markov chain $y(t)$ in $k-$th consecutive state. Condition on the sequence of the chain states $y_k,\, k \geq 0$, the joint distribution of the random variables $\tau_k,\, k \geq 0$, coincides with the joint distribution of independent random variables exponentially distributed with parameters $\sigma(y_k),\, k = 0, 1, \ldots, n$, so the transition probabilities of the chain $y(t)$ are

$$p_{yw}(t) = \sum_n \sum_{(y \to w)} \mathsf{P}\{(y \to w)\} \int_{\Delta_n^t} e^{-\sigma(w)(t - t_n)} \prod_{k=1}^n \sigma(y_{k-1}) e^{-\sigma(y_{k-1})(t_k - t_{k-1})}\, dt_1 \ldots dt_n,$$

where n corresponds to the number of jumps of the chain y during the time interval $[0, t]$, the inner sum is taken over all trajectories $(y \to w) = \{y = y_0, y_1, \ldots, y_n = w\}$ with n jumps, integration is taken over $\Delta_n^t = \{0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq t\}$, and

$$\mathsf{P}\{(y \to w)\} = p(y, y_1) p(y_1, y_2) \cdots p(y_{n-1}, w)$$

is a probability of the corresponding path for the embedded chain. The equation (30) implies that the integrand in $p_{yw}(t)$ is uniformly bounded from below by the expression

$$(\sigma_{*,N})^n \exp(-\sigma_N^* t_1) \cdots \exp(-\sigma_N^*(t_n - t_{n-1})) \exp(-\sigma_N^*(t - t_n)),$$

and, hence,

$$\int_{\Delta_t^n} e^{-\sigma(w)(t-t_n)} \prod_{k=1}^n \sigma(y_{k-1}) e^{-\sigma(y_{k-1})(t_k - t_{k-1})} \, dt_1 \dots dt_n \geq \frac{(\sigma_{*,N} t)^n}{n!} e^{-\sigma_N^* t}$$

$$= \mathsf{P} \left\{ \Pi_t = n \right\} e^{-\left(\sigma_N^* - \sigma_{*,N} \right) t},$$

where Π_t is a Poisson process with parameter $\sigma_{*,N}$. It provides us with a lower bound for the transition probabilities of the time-continuous chain:

$$p_{yz_0}(t) \geq \left(\sum_n p^n(y, z_0) \mathsf{P} \left\{ \Pi_t = n \right\} \right) e^{-\left(\sigma_N^* - \sigma_{*,N} \right) t}.$$

It is easy now to get a lower bound for probabilities $p^n(y, z_0)$. Fix some $z_0 \in S$ and denote $\xi = \pi^Y(z_0)$. It follows from _ergodicity_ of the chain Y, that for any fixed z_0

$$p^m(y, z_0) \geq \pi^Y(z_0)/2 = \xi/2 > 0,$$

for all $m \geq m_1 = m_1(y)$. For a _Doeblin_ Markov chain we have more strong conclusion, namely, the above number m_1 does not depend on y. Let us fix such m_1 and show that the continuous-time Markov chain $y(t)$ satisfies to the Doeblin condition. Indeed,

$$\begin{aligned} p_{yz_0}(t) & \geq \left(\sum_{n < m_1} p^n(y, z_0) \mathsf{P} \left\{ \Pi_t = n \right\} + \sum_{n \geq m_1} p^n(y, z_0) \mathsf{P} \left\{ \Pi_t = n \right\} \right) e^{-\left(\sigma_N^* - \sigma_{*,N} \right) t} \\ & \geq \sum_{n \geq m_1} p^n(y, z_0) \mathsf{P} \left\{ \Pi_t = n \right\} e^{-\left(\sigma_N^* - \sigma_{*,N} \right) t} \\ & \geq \frac{\xi}{2} \mathsf{P} \left\{ \Pi_t \geq m_1 \right\} e^{-\left(\sigma_N^* - \sigma_{*,N} \right) t}. \end{aligned}$$

Hence, the Doeblin condition holds: we choose any z_0 as j_0, take a corresponding m_1, fix any $h > 0$ and put

$$\delta = \frac{\xi}{2} \, \mathsf{P} \left\{ \Pi_h \geq m_1 \right\} e^{-\left(\sigma_N^* - \sigma_{*,N} \right) h}.$$

Proof of the theorem is over.

4.2   Evolution of the center of mass

Consider the following function on the state space $\mathbf{Z}^N$: $m(x_1, \dots, x_N) = (x_1 + \dots + x_N)/N$. So if each particle has the mass 1 and $x_1(t), \dots, x_N(t)$ are positions of particles, then $m(x_1(t), \dots, x_N(t))$

is the *center of mass* of the system. We are interested in evolution of $\mathsf{E}\, m(x_1(t), \ldots, x_N(t))$. A direct calculation gives that

$$
\begin{aligned}
(G_N m)(x_1, \ldots, x_N) &= \sum_{i=1}^{N} \left(\frac{\alpha}{N} - \frac{\beta}{N} \right) + \sum_{i=1}^{N} \sum_{j \neq i} \left(-\frac{x_i - x_j}{N} \right) I_{\{x_i > x_j\}} \frac{\mu_N}{N} \\
&= (\alpha - \beta) - \frac{\mu_N}{N^2} \sum_{i<j} |x_i - x_j|,
\end{aligned} \tag{35}
$$

where we have used the following equalities

$$
\begin{aligned}
m\left(x \pm e_i^{(N)} \right) - m(x) &= \pm \frac{1}{N}, \\
m\left(x - (x_i - x_j) e_i^{(N)} \right) - m(x) &= -\frac{x_i - x_j}{N}, \\
(x_i - x_j) I_{\{x_i > x_j\}} + (x_j - x_i) I_{\{x_j > x_i\}} &= |x_i - x_j|.
\end{aligned}
$$

Note that the summand

$$
-\frac{\mu_N}{N^2} \sum_{i<j} |x_i - x_j|
$$

added by the interaction to the "free dynamics" drift $(\alpha - \beta)$ depends only on the relative disposition of particles. So the center of mass of the system moves with speed which tends to the value

$$
(\alpha - \beta) - \mu_N \frac{N-1}{2N} \mathsf{E}_\pi |x_1 - x_2|
$$

as t goes to infinity. Here $\mathbf{E}_\pi |x_1 - x_2|$ is the mean distance between two particles calculated with respect to the stationary measure π of the Markov chain Y.

Using this fact and Theorem 2.2 we can describe the *long time behavior of the particle system* in the initial coordinates x's as follows. Theorem 2.2 means that the system of stochastic interacting particles possesses some relative stability. In coordinates y the system approaches exponentially fast its equilibrium state. In the meantime the particle system considered as a "single body" moves with an asymptotically constant speed. The speed differs from the mean drift of the free particle motion and this difference is due to the interaction between the particles.

5 On travelling waves and long time evolution of solution of PDE

We deal here with partial differential equation in variables $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$.

Definition 5.1 *Function $w = w(x)$ is called a travelling wave solution of some PDE if there exists $v \in \mathbf{R}$ such that the function $u(t, x) = w(x - vt)$ is a solution of this PDE. The number v is speed of the wave w.*

We are interested only in the travelling waves having the following properties: **U1)** $w(x) \in [0, 1]$; **U2)** $w(x)$ and $dw(x)/dx$ have limits as $x \to \pm\infty$ and, besides, $w(-\infty) = 1$ and $w(+\infty) = 0$. We identify two travelling waves $w_1(x)$ and $w_2(x)$ if $w_1(x) = w_2(x - c)$ for some c.

For any probabilistic solution $0 \leq u(t, x) \leq 1$ we define a function $r(t)$ such that $u(t, r(t)) \equiv \frac{1}{2}$. Let a function $w(x)$ be a travelling wave solution. Without loss of generality we can assume that $w(0) = 1/2$.

Definition 5.2 *A solution $u(t, x)$ converges* in form *to the travelling wave $w(x)$ if*

$$\lim_{t \to +\infty} u(t, x + r(t)) = w(x),$$

uniformly on any finite interval. The solution $u(t, x)$ converges in speed *to the travelling wave $w(x)$ if there exists $r'(t) = dr(t)/dt$ and $\lim_{t \to +\infty} r'(t) = v$, where v is a speed of the travelling wave $w(x)$.*

First order equation.   It is easy to check for the equation (4) for every $v < \lambda$ there exists a unique (up to shift) travelling wave solution having properties U1–U2 and this travelling wave solution is given by the following formula $w_v(x) = \left(1 + \exp\left(\dfrac{\mu}{\lambda - v} x \right) \right)^{-1}$.

Proposition 5.1 *If for some $C > 0, \nu > 0$, the initial profile $\psi(x) \in H(\mathbf{R})$ of the equation (4) has the following asymptotic behavior $1 - \psi(x) \sim C \exp(\nu x)$, as $x \to -\infty$, then there exists $x_0 \in \mathbf{R}$ such that for every $x \in \mathbf{R}$*

$$|u(t, x) - w_v(x - x_0 - vt)| \to 0,$$

as $t \to \infty$, where $v = \lambda - \mu/\nu$.

The proof of Proposition 5.1 is a direct calculation based on the exact formula (29).

The formula (28) yields that $u^\circ(t_1, x) \geq u^\circ(t_2, x)$ for any $t_1 < t_2$ and this observation immediately implies the following statement: for every x $u^\circ(t, x) \to I_{\{y: \psi(y)=1\}}(x)$, as $t \to \infty$. As a direct application of this property we can obtain the following

Proposition 5.2 *Assume that the initial profile has the form $\psi(x) = I_{\{y<b\}}(x)$ for some $b \in \mathbf{R}$. Then*

$$|u(t, x) - I_{\{y<b\}}(x - \lambda t)| \to 0, \qquad t \to \infty.$$

So the function $w(x) = I_{\{y\leq 0\}}(x)$ is a unique (up to shift) non-increasing continuous from the right travelling wave corresponding to the maximal possible speed $v = \lambda$. It is easy to see that this function is a limiting case of $w_v(x)$ as $v \to \lambda - 0$.

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Second order equation. The existence of travelling waves for parabolic partial differential equations was a subject of studying in many papers followed to the paper [9]. A review of many results can be found in [24] (see also [20]) and for completeness of the text we mention some of them. Reformulating the well-known results ([24]) we obtain that travelling waves of the equation (5) can move only from the right to the left. It means that the speed of any travelling wave is negative and, moreover, is bounded away from 0.

Proposition 5.3 *For equation (5) for every $v \leq v_* = -\sqrt{4\gamma\mu}$ there exist and unique (up to shift) travelling wave solution with speed v. There are no other travelling wave solutions satisfying the conditions U1 and U2.*

If a function $f = f(x)$ is such that $f(x) \leq 1$, $f(x) \to 1$ as $x \to -\infty$ and there exists a limit $\varkappa = \lim_{x \to -\infty} x^{-1}\log(1 - f(x)) > 0$, then the number $\varkappa$ is called *Lyapunov exponent of the function f (at minus infinity).* It is well known ([24]) that for the equation (5) a travelling $w(x)$ with speed v has the following Lyapunov exponent at minus infinity

$$\varkappa(v) = \left(-v - \sqrt{v^2 - 4\gamma\mu}\right) / (2\gamma).$$

Hence we get that for the travelling wave with minimal in absolute value speed $v_* = -\sqrt{4\gamma\mu}$ the Lyapunov exponent is $\varkappa(v_*) = \sqrt{\mu/\gamma}$.

Proposition 5.4 ([24]) *Assume that an initial function $\psi(x)$ has a Lyapunov exponent $\varkappa$. Then*
a) if $\varkappa \geq \sqrt{\mu/\gamma}$ then the solution $u(t, x)$ of the problem (5) converges in form and in speed to the travelling wave moving with the minimal speed $v_* = -\sqrt{4\gamma\mu}$;
b) if $\varkappa < \sqrt{\mu/\gamma}$ then the solution $u(t, x)$ of the problem (5) converges in form and in speed to the travelling wave with speed $v = -\left(\gamma\varkappa + \dfrac{\mu}{\varkappa}\right)$, or, in other words, $\varkappa(v) = \varkappa$.

We see from the above analysis that both the first order PDE (4) and the second order PDE (5) exhibit similar long-time behavior of their solutions. This seems very natural if we recall from Theorem 2.1 that the both equations arise as hydrodynamical approximations of the same stochastic particle system.

A Appendix

A.1 Strong topology on the Skorokhod space

Remind that Schwartz space $S(\mathbf{R})$ is a Frechet space (see [18]). In the dual space $S'(\mathbf{R})$ of tempered distributions there are at least two ways to define topology (both not metrizable):

1) *weak topology* on $S'(\mathbf{R})$, where all functionals $(\,\cdot\,,\phi)$, $\quad \phi \in S(\mathbf{R})$ are continuous.

2) *strong topology* (*s.t.*) on $S'(\mathbf{R})$, which is generated by the set of seminorms

$$\left\{ \rho_A(M) = \sup_{\phi \in A} |(M,\phi)| \ : \ A \subset S(\mathbf{R}) \ - \ bounded \right\}.$$

Below we shall consider $S'(\mathbf{R})$ as equipped with the strong topology. The problem of introducing of the Skorokhod topology on the space $D_T(S') := D([0,T], S'(\mathbf{R}))$ was studied in [15] and [7]. We follow these papers. For each seminorm ρ_A on $S'(\mathbf{R})$ we define the following pseudometric on $D([0,T], S'(\mathbf{R}))$

$$d_A(y,z) = \inf_{\lambda \in \Lambda} \left\{ \sup_t \rho_A(y_t - z_{\lambda(t)}) + \sup_{t \neq s} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right\}, \quad y,z \in D([0,T], S'(\mathbf{R})),$$

where $\inf$ is taken over the set $\Lambda = \{\lambda = \lambda(t),\, t \in [0,T]\}$ of all strictly increasing continuous mappings of $[0,T]$ onto itself. Introducing on $D([0,T], S'(\mathbf{R}))$ the projective limit topology of $\{d_A(\cdot,\cdot)\}$ we get a completely regular topological space.

A.2 Mitoma theorem

Let $\mathcal{B}_{D_T(S')}$ be the corresponding Borel σ-algebra. Let $\{P_n\}$ be a sequence of probability measures on $\big(D_T(S'), \mathcal{B}_{D_T(S')}\big)$. For each $\phi \in S(\mathbf{R})$ consider a map $\mathcal{I}_\phi : y \in D_T(S') \to (y,\phi) \in D_T(\mathbf{R})$. The following result belongs to I. Mitoma [15].

Theorem A.1 *Suppose that for any $\phi \in S(\mathbf{R})$ the sequence $\big\{ P_n \mathcal{I}_\phi^{-1} \big\}$ is tight in $D_T(\mathbf{R})$. Then the sequence $\{P_n\}$ itself is tight in $D_T(S')$.*

A.3 Probability measures on the Skorokhod space: tightness

Let $\{(\xi_t^n, t \in [0,T])\}_{n \in \mathbf{N}}$ be a sequence of real random processes which trajectories are right-continuous and admit left-hand limits for every $0 < t \leq T$. We will consider ξ^n as random elements with values in the Skorokhod space $D_T(\mathbf{R}) := D\big([0,T], \mathbf{R}^1\big)$ with the standard topology. Denote P_T^n the distribution of ξ^n, defined on the measurable space $(D_T(\mathbf{R}), \mathcal{B}\,(D_T(\mathbf{R})))$. The following result can be found in [1].

Theorem A.2 *The sequence of probability measures $\{P_T^n\}_{n \in \mathbf{N}}$ is tight iff the following two conditions hold:*

1) for any $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$\sup_n P_T^n \left(\sup_{0 \leq t \leq T} |\xi_t^n| > C(\varepsilon) \right) \leq \varepsilon\,;$$

2) for any $\varepsilon > 0$

$$\lim_{\gamma \to 0} \limsup_n P_T^n \left(\xi. : \ w'(\xi; \gamma) > \varepsilon \right) = 0 \,,$$

where for any function $f : [0, T] \to \mathbf{R}$ and any $\gamma > 0$ we define

$$w'(f; \gamma) = \inf_{\{t_i\}_{i=1}^r} \max_{i < r} \sup_{t_i \le s < t < t_{i+1}} |f(t) - f(s)| \,,$$

moreover the inf *is over all partitions of the interval $[0, T]$ such that*

$$0 = t_0 < t_1 < \cdots < t_r = T, \qquad t_i - t_{i-1} > \gamma, \quad i = 1, \ldots, r.$$

The following theorem is known as the sufficient condition of Aldous [8, Proposition 1.6].

Theorem A.3 *Condition 2) of the previous theorem follows from the following condition*

$$\forall \varepsilon > 0 \qquad \lim_{\gamma \to 0} \limsup_n \sup_{\tau \in \mathcal{R}_T, \, \theta \le \gamma} P_T^n \left(|\xi_{\tau+\theta} - \xi_\tau| > \varepsilon \right) = 0 \,,$$

where $\mathcal{R}_T$ is the set of Markov moments (stopping times) not exceeding T.

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